LEPTON POLARIZATION IN NEUTRINO–NUCLEON INTERACTIONS

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Abstract

We derive generic formulas for the polarization density matrix of leptons produced in νN and $\overline{\nu} N$ collisions and briefly consider some important particular cases. Next we employ the general formalism in order to include the final lepton mass and spin into the popular model by Rein and Sehgal for single pion neutrinoproduction.

1 Introduction

Polarization of leptons generated in νN and $\overline{\nu} N$ collisions is important for studying neutrino oscillations and relevant phenomena in experiments with atmospheric and accelerator neutrino beams. Let us shortly touch upon a few illustrative examples.

• Contained τ lepton events provide the primary signature for $\nu_\mu \to \nu_\tau$ oscillations. Besides they are a source of unavoidable background to the future proton decay experiments. But a low or intermediate energy τ lepton generated inside a water Cherenkov detector is unobservable in itself and may only be identified through the τ decay secondaries whose momentum configuration is determined by the τ lepton helicity.

• In case of $\nu_\mu - \nu_\tau$ mixing, the leptonic decays of τ's generated inside the Earth yield an extra contribution into the flux of through-going upward-going muons (TUM) and stopping muons (SM) which is absent in case of $\nu_\mu - \nu_s$ or $\nu_\mu - \nu_e$ mixing. The absolute value and energy spectrum of the "$\tau_{\mu 3}$ muons" are affected by the τ beam polarization. The contribution is evidently small but measurable in future large-scale experiments, particularly those with magnetized tracking calorimeters (like in the experiments NuMI–MINOS and MONOLITH). Note that the energy and angular distributions of the charge ratio for the "$\tau_{\mu 3}$ muons" are considerably differ from those for the "direct" TUM and SM since the longitudinal polarizations of τ^+ and τ^- have in average opposite signs.

• Decay of ν_μ or $\overline{\nu}_\mu$ induced muons with energy below the detection threshold may produce detectable electrons whose energy distributions are affected by the muon polarization. Such events, being classified as "e-like" (for a water detector) or "showering" (for an iron detector), mimic the ν_e or $\overline{\nu}_e$ induced events.

In this paper, we derive general formulas for the lepton polarization density matrix by applying a covariant method (Sect. 2) and briefly consider their applications to deep inelastic, quasielastic and resonance neutrino interactions. We explicitly demonstrate that the perpendicular and transverse polarizations are dependent of an intrinsically indeterminate phase and thus unobservable in contrast with the longitudinal polarization and degree of polarization. In Sect. 3 we discuss with some details a generalization of the Rein–Sehgal model for single pion neutrinoproduction through baryon resonances which takes into account the final lepton mass and spin.

2 Polarization density matrix

The lepton polarization vector $\boldsymbol{\mathcal{P}} = (\mathcal{P}_P, \mathcal{P}_T, \mathcal{P}_L)$ is defined through the polarization density matrix $\boldsymbol{\rho} = \frac{1}{2}(1 + \boldsymbol{\sigma}\boldsymbol{\mathcal{P}})$ whose matrix elements are given by contracting the leptonic tensor $L^{\alpha\beta}_{\lambda\lambda'}$ with the spin-averaged hadronic tensor $W_{\alpha\beta}$. The leptonic tensor is given by

$$
L^{\alpha\beta}_{\lambda\lambda'} = \begin{cases} j^{\alpha}_{\lambda}\left(j^{\beta}_{\lambda'}\right)^{*} & \text{with } j^{\alpha}_{\lambda} = \overline{u}\left(k', s\right)\gamma^{\alpha}\left(\dfrac{1 - \gamma_5}{2}\right)u(k) \ \text{ for } \nu_{\ell}, \\[3mm] \overline{j}^{\alpha}_{\lambda}\left(\overline{j}^{\beta}_{\lambda'}\right)^{*} & \text{with } \overline{j}^{\alpha}_{\lambda} = \overline{v}(k)\gamma^{\alpha}\left(\dfrac{1 - \gamma_5}{2}\right)v\left(k', s\right) \ \text{ for } \overline{\nu}_{\ell}, \end{cases}
\tag{1}
$$

where k and k' are the 4-momenta of ν_{ℓ} or $\overline{\nu}_{\ell}$ and lepton ℓ^- or ℓ^+ ($\ell = e, \mu, \tau$), λ and λ' are the lepton helicities, s is the axial 4-vector of the lepton spin.

It can be shown that the weak leptonic currents j^{α}_{λ} and $\overline{j}^{\alpha}_{\lambda}$ are expressed as

$$
j^{\alpha}_{\lambda} = N_{\lambda}\left[k'^{\alpha}(ks) - s^{\alpha}(kk') - i\epsilon^{\alpha\beta\gamma\delta}s_{\beta}k_{\gamma}k'_{\delta} + m_{\ell}k^{\alpha}\right],
\tag{2a}
$$

$$
\overline{j}^{\alpha}_{\lambda} = \lambda N_{\lambda}\left[k'^{\alpha}(ks) - s^{\alpha}(kk') + i\epsilon^{\alpha\beta\gamma\delta}s_{\beta}k_{\gamma}k'_{\delta} - m_{\ell}k^{\alpha}\right].
\tag{2b}
$$

Here m_{ℓ} is the lepton mass and the normalization constant N_{λ} is expressed in terms of the kinematic variables and of two intrinsically indeterminate phases φ_+ and φ_-:

$$
N_{\lambda} = \frac{\exp\left(\pm i\varphi_{\lambda}\right)}{\sqrt{(kk') \pm m_{\ell}(ks)}} = \frac{\exp\left(\pm i\varphi_{\lambda}\right)}{m_{\ell}}\sqrt{\frac{E_{\ell} \mp \lambda P_{\ell}}{E_{\nu}\left(1 \mp \lambda\cos\theta\right)}}.
\tag{3}
$$

Here E_{ν}, E_{ℓ} and θ are, respectively, the incident neutrino energy, lepton energy and the scattering angle of the lepton in the lab. frame, $P_{\ell} = \sqrt{E_{\ell} - m_{\ell}^2}$; the upper (lower) signs are for ν_{ℓ} ($\overline{\nu}_{\ell}$). As it follows from Eq. (2a), the components of the neutrino current are

$$
\begin{aligned}
j^0_{\lambda} &= \exp\left(i\varphi_{\lambda}\right)\sqrt{\left(1 - \lambda\cos\theta\right)E_{\nu}\left(E_{\ell} - \lambda P_{\ell}\right)}, \\
\mathbf{j}_{\lambda} &= \frac{\lambda m_{\ell}N_{\lambda}}{P_{\ell}}\left[\lambda P_{\ell}\mathbf{k} - E_{\nu}\mathbf{k}' + \mathbf{i}\left(\mathbf{k} \times \mathbf{k}'\right)\right].
\end{aligned}
\tag{4}
$$

As it follows from Eqs. (2) and (3), the currents for neutrino and antineutrino are related by

$$
\overline{j}^{\alpha}_{\lambda} = -\lambda\left(j^{\alpha}_{-\lambda}\right)^{*}.
\tag{5}
$$

We use the standard representation for the hadronic tensor (see, e.g., Ref. [1])

$$
\begin{aligned}
W_{\alpha\beta} = &-g_{\alpha\beta}\,W_1 + \frac{p_{\alpha}\,p_{\beta}}{M^2}\,W_2 - \frac{i\,\epsilon_{\alpha\beta\rho\sigma}\,p^{\rho}\,q^{\sigma}}{2M^2}\,W_3 \\
&+ \frac{q_{\alpha}\,q_{\beta}}{M^2}\,W_4 + \frac{p_{\alpha}\,q_{\beta} + q_{\alpha}\,p_{\beta}}{2M^2}\,W_5 + i\,\frac{p_{\alpha}\,q_{\beta} - q_{\alpha}\,p_{\beta}}{2M^2}\,W_6
\end{aligned}
\tag{6}
$$

which includes 6 nucleon structure functions, W_n, whose explicit form is defined by the particular subprocess (QE, RES or DIS). Here p and M are the nucleon 4-momentum and mass, respectively, $q = k - k'$ is the W boson 4-momentum. By applying Eqs. (1), (2) and (6), we obtain

$$
\rho_{\lambda\lambda'} = \frac{L^{\alpha\beta}_{\lambda\lambda'}W_{\alpha\beta}}{4ME_{\nu}\mathcal{R}} = \frac{m_{\ell}^2 E_{\nu}N_{\lambda}N^{*}_{\lambda'}}{4M\mathcal{R}}\sum_{k=1}^{6}A^{k}_{\lambda\lambda'}W_k,
$$

$$A^1_{\lambda\lambda'} = 2\left(\eta_{\lambda\lambda'} \mp \eta_{-\lambda\lambda'}\right)\sin^2\theta,$$

$$A^2_{\lambda\lambda'} = 4\left(\eta_{\pm\lambda}\eta_{\pm\lambda'}\sin^4\frac{\theta}{2} + \eta_{\mp\lambda}\eta_{\mp\lambda'}\cos^4\frac{\theta}{2}\right) \pm \eta_{-\lambda\lambda'}\sin^2\theta,$$

$$A^3_{\lambda\lambda'} = \pm\sin^2\theta\left(\eta_{\pm\lambda}\eta_{\pm\lambda'}\frac{E_\nu - P_\ell}{M} + \eta_{\mp\lambda}\eta_{\mp\lambda'}\frac{E_\nu + P_\ell}{M} \mp \eta_{-\lambda\lambda'}\frac{E_\nu}{M}\right),$$

$$A^4_{\lambda\lambda'} = 4\left[\eta_{\pm\lambda}\eta_{\pm\lambda'}\frac{(E_\nu + P_\ell)^2}{M^2}\sin^4\frac{\theta}{2} + \eta_{\mp\lambda}\eta_{\mp\lambda'}\frac{(E_\nu - P_\ell)^2}{M^2}\cos^4\frac{\theta}{2}\right]$$
$$\pm \eta_{-\lambda\lambda'}\frac{m_\ell^2}{M^2}\sin^2\theta,$$

$$A^5_{\lambda\lambda'} = 4\left[\eta_{\pm\lambda}\eta_{\pm\lambda'}\frac{E_\nu + P_\ell}{M}\sin^4\frac{\theta}{2} + \eta_{\mp\lambda}\eta_{\mp\lambda'}\frac{E_\nu - P_\ell}{M}\cos^4\frac{\theta}{2}\right]$$
$$\mp \eta_{-\lambda\lambda'}\frac{E_\ell}{M}\sin^2\theta,$$

$$A^6_{\lambda\lambda'} = i\left(\frac{\lambda - \lambda'}{2}\right)\frac{P_\ell}{M}\sin^2\theta,$$

where $\eta_\lambda \equiv (1 + \lambda)/2$ ($\eta_+ = 1$, $\eta_- = 0$) and the dimensionless normalization factor $\mathcal{R}$ is given by the condition $\mathrm{Tr}\,\rho = 1$:

$$\mathcal{R} = \frac{1}{4ME_\nu}\left(L^{\alpha\beta}_{++} + L^{\alpha\beta}_{--}\right)W_{\alpha\beta}$$
$$= \left(\frac{E_\ell - P_\ell\cos\theta}{M}\right)\left(W_1 + \frac{m_\ell^2}{2M^2}W_4\right) + \left(\frac{E_\ell + P_\ell\cos\theta}{2M}\right)W_2$$
$$\pm\left[\left(\frac{E_\nu + E_\ell}{M}\right)\left(\frac{E_\ell - P_\ell\cos\theta}{2M}\right) - \frac{m_\ell^2}{2M^2}\right]W_3 - \frac{m_\ell^2}{2M^2}W_5.$$

Now we can find the explicit formulas for the elements of the polarization density matrix in terms of variables E_ν, P_ℓ and θ:

$$\rho_{++}(E_\nu, P_\ell, \theta) = \rho_{--}(E_\nu, -P_\ell, \pi - \theta) = \frac{E_\ell \mp P_\ell}{2M\mathcal{R}}\mathcal{Z},$$

$$\rho_{+-}(E_\nu, P_\ell, \theta) = \rho^*_{-+}(E_\nu, P_\ell, \theta) = \frac{m_\ell\sin\theta}{4M\mathcal{R}}(\mathcal{X} - i\mathcal{Y})e^{i\varphi}.$$

Here we have introduced the following notation:

$$\mathcal{X} = \mp\left(2W_1 - W_2 - \frac{m_\ell^2}{M^2}W_4 + \frac{E_\ell}{M}W_5\right) - \frac{E_\nu}{M}W_3,$$

$$\mathcal{Y} = -\frac{P_\ell}{M}W_6,$$

$$\mathcal{Z} = (1 \pm \cos\theta)\left(W_1 \pm \frac{E_\nu \mp P_\ell}{2M}W_3\right)$$
$$+ \frac{1 \mp \cos\theta}{2}\left[W_2 + \frac{E_\ell \pm P_\ell}{M}\left(\frac{E_\ell \pm P_\ell}{M}W_4 - W_5\right)\right],$$

and $\varphi = \varphi_+ - \varphi_-$. Finally the projections of the lepton polarization vector are given by

$$\begin{pmatrix}\mathcal{P}_P \\ \mathcal{P}_T\end{pmatrix} = \frac{m_\ell\sin\theta}{2M\mathcal{R}}\begin{pmatrix}\cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\mathcal{X} \\ \mathcal{Y}\end{pmatrix}, \tag{7a}$$

3

$$\mathcal{P}_L = \mp 1 \pm \frac{m_\ell^2}{M^2 \mathcal{R}} \left\{ \left[\left(\frac{2M}{E_\ell + P_\ell} \right) W_1 \pm \left(\frac{E_\nu - P_\ell}{E_\ell + P_\ell} \right) W_3 \right] \cos^2 \frac{\theta}{2} \right.$$
$$\left. + \left[\left(\frac{M}{E_\ell + P_\ell} \right) W_2 + \left(\frac{E_\ell + P_\ell}{M} \right) W_4 - W_5 \right] \sin^2 \frac{\theta}{2} \right\}. \tag{7b}$$

By putting $\varphi = 0$,[1] the formulas for $\mathcal{P}_P$ and $\mathcal{P}_L$ exactly coincide with those of Ref. [2] (obtained within a noncovariant approach under assumption $W_6 = 0$).

Several simple conclusions immediately follow from Eqs. (7). First, the perpendicular and transverse projections are unobservable quantities in contrast with the longitudinal projection of $\mathcal{P}$ and the degree of polarization $|\mathcal{P}|$. Second, supposing that $W_6 = 0$ (as is the case in the Standard Model) one can force the polarization vector to lie in the production plane. Third, a massless lepton is fully polarized, $\mathcal{P} = (0, 0, \mp 1)$. In particular, at the energies of our interest, electron is always fully polarized while in general, this is not the case for muon and τ lepton.

We end this section with a short description of the structure functions relevant to the three fundamental subprocesses, DIS, QE and RES.

2.1 Deep inelastic scattering (DIS)

In this case the relation between the structure functions $W_n^{(\mathrm{DIS})}(x, Q^2)$ and measurable quantities $F_n(x, Q^2)$ is obtained in a straightforward manner:

$$W_1^{(\mathrm{DIS})}(x, Q^2) = F_1(x, Q^2), \quad W_n^{(\mathrm{DIS})}(x, Q^2) = w^{-1} F_n(x, Q^2), \quad n = 2, \ldots, 6.$$

Here $Q^2 = -q^2$, $x = Q^2/2(pq)$ is the Bjorken scaling variable and $w = (pq)/M^2$.

The generally accepted relations between the functions F_1, F_2, F_4 and F_5 are

$$F_1 = F_5 = \frac{F_2}{2x(1 + R)} \left(1 + \frac{x^2}{x'} \right), \quad F_4 = \frac{1}{2x} \left(\frac{F_2}{2x} - F_1 \right),$$

where R is the ratio of longitudinal to transverse cross sections in DIS and $x' = Q^2/(4M^2)$.

2.2 Quasielastic scattering (QE)

The charged hadronic currents describing the QE processes are written [1]

$$\langle p, p' | \widehat{J}_\alpha^+ | n, p \rangle = \cos \theta_C \overline{u}_p(p') \, \Gamma_\alpha \, u_n(p),$$
$$\langle n, p' | \widehat{J}_\alpha^- | p, p \rangle = \cos \theta_C \overline{u}_n(p') \, \overline{\Gamma}_\alpha \, u_p(p),$$

where θ_C is the Cabibbo mixing angle, $p' = p + q$ and the vertex function

$$\Gamma_\alpha = \gamma_\alpha F_V + i \sigma_{\alpha\beta} \frac{q_\beta}{2M} F_M + \frac{q_\alpha}{M} F_S + \left(\gamma_\alpha F_A + \frac{p_\alpha + p'_\alpha}{M} F_T + \frac{q_\alpha}{M} F_P \right) \gamma_5,$$

is defined through the six, in general complex, form factors $F_i(q^2)$. A standard calculation then yields

$$W_n^{(\mathrm{QE})}(x, Q^2) = \cos^2 \theta_C w^{-1} \omega_n(Q^2) \delta(1 - x), \quad n = 1, \ldots, 6,$$

---

[1]We adopt this convention from here on. Therefore, according to Eq. (7a), $\mathcal{P}_P \propto \mathcal{X}$ and $\mathcal{P}_T \propto \mathcal{Y}$.

where the functions ω_n are the bilinear combinations of the form factors:

$$
\begin{aligned}
\omega_1 &= |F_A|^2 + x' \left(|F_A|^2 + |F_V + F_M|^2 \right), \\
\omega_2 &= |F_V|^2 + |F_A|^2 + x' \left(|F_M|^2 + 4\,|F_T|^2 \right), \\
\omega_3 &= -2\,\mathrm{Re}\left[F_A^* \left(F_V + F_M \right) \right], \\
\omega_4 &= \mathrm{Re}\left[F_V^* \left(F_S - \tfrac{1}{2}F_M \right) - F_A^* \left(F_T + F_P \right) \right] + x' \left(\tfrac{1}{2}\,|F_M - F_S|^2 + |F_T + F_P|^2 \right) \\
&\quad - \tfrac{1}{4}\left(1 + x' \right) |F_M|^2 + \left(1 + \tfrac{1}{2}x' \right) |F_S|^2, \\
\omega_5 &= 2\,\mathrm{Re}\left[F_S^* \left(F_V - x' F_M \right) - F_T^* \left(F_A - 2x' F_P \right) \right] + \omega_2, \\
\omega_6 &= 2\,\mathrm{Im}\left[F_S^* \left(F_V - x' F_M \right) + F_T^* \left(F_A - 2x' F_P \right) \right].
\end{aligned}
$$

The only difference between this result and that from Ref. [1] is in the relative sign of the terms in ω_6.[2] Assuming all the form factors to be real we have $\omega_6 = 0$ and thus $\mathcal{P}_T = 0$.

2.3 Single resonance production (RES)

Let us now consider the case of single Δ resonance neutrinoproduction,

$$
\nu_\ell + n(p) \rightarrow \ell^- + \Delta^+ \left(\Delta^{++} \right), \quad \overline{\nu}_\ell + n(p) \rightarrow \ell^+ + \Delta^- \left(\Delta^0 \right).
$$

Assuming the isospin symmetry and applying the Wigner-Eckart theorem, the hadronic weak current matrix elements are given by [2]

$$
\begin{aligned}
\langle \Delta^+,\, p'|\widehat{J}_\alpha|n,\, p \rangle &= \langle \Delta^0,\, p'|\widehat{J}_\alpha|p,\, p \rangle = \cos\theta_C \overline{\psi}^\beta \left(p' \right) \Gamma_{\alpha\beta}\, u(p), \\
\langle \Delta^{++},\, p'|\widehat{J}_\alpha|p,\, p \rangle &= \langle \Delta^-,\, p'|\widehat{J}_\alpha|n,\, p \rangle = \sqrt{3}\,\cos\theta_C \overline{\psi}^\beta \left(p' \right) \Gamma_{\alpha\beta}\, u(p).
\end{aligned}
$$

Here $\psi^\alpha \left(p' \right)$ is the Rarita-Schwinger spin-vector for Δ resonance, $u(p)$ is the Dirac spinor for neutron or proton and the vertex tensor $\Gamma_{\alpha\beta}$ is expressed in terms of the eight weak transition form factors $C^{V,A}_{3,4,5,6} \left(Q^2 \right)$ [3]:

$$
\begin{aligned}
\Gamma_{\alpha\beta} &= \left[C_3^V \frac{g_{\alpha\beta}\widehat{q} - \gamma_\alpha q_\beta}{M} + C_4^V \frac{g_{\alpha\beta}\left(qp' \right) - p'_\alpha q_\beta}{M^2} + C_5^V \frac{g_{\alpha\beta}\left(qp \right) - p_\alpha q_\beta}{M^2} + C_6^V \frac{q_\alpha q_\beta}{M^2} \right] \gamma_5 \\
&\quad + C_3^A \frac{g_{\alpha\beta}\widehat{q} - \gamma_\alpha q_\beta}{M} + C_4^A \frac{g_{\alpha\beta}\left(qp' \right) - p'_\alpha q_\beta}{M^2} + C_5^A g_{\alpha\beta} + C_6^A \frac{q_\alpha q_\beta}{M^2}.
\end{aligned}
$$

Below we will assume that time-reversal invariance holds so that all the form factors are relatively real. After accounting for the explicit form of a spin 3/2 projection operator and computing the proper convolutions we arrive at the following expressions for $W_n^{\text{(RES)}}$:

$$
\begin{aligned}
W_n^{\text{(RES)}} &= \kappa \cos^2\theta_C M M_\Delta \left| \eta_{\text{BW}}^\Delta(W) \right|^2 \sum_{j,k=3\text{ to }6} \left(V_n^{jk} C_j^V C_k^V + A_n^{jk} C_j^A C_k^A \right), \quad n \neq 3, \\
W_3^{\text{(RES)}} &= 2\kappa \cos^2\theta_C M M_\Delta \left| \eta_{\text{BW}}^\Delta(W) \right|^2 \sum_{j,k=3\text{ to }6} K_n^{jk} C_j^V C_k^A \quad \text{and} \quad W_6^{\text{(RES)}} = 0.
\end{aligned}
$$

(8)

---

[2]According to Llewellyn Smith, the functions $\omega_5' = \omega_5 - \omega_2$ and ω_6 are, respectively, the real and imaginary parts of a unique function. Our examination does not confirm this property for the general case of nonvanishing second-class current induced form factors F_S and F_T.

Here $\kappa = 2/3$ for Δ^+ and Δ^0 production or $\kappa = 2$ for Δ^{++} and Δ^- production;

$$\left|\eta_{\mathrm{BW}}^{\Delta}(W)\right|^2 = \frac{1}{2\pi}\left[\frac{\Gamma_\Delta(W)}{\left(W^2 - M_\Delta^2\right)^2 + \Gamma_\Delta^2(W)/4}\right]$$

is the Breit-Wigner factor and $\Gamma_\Delta(W)$ is the running width of the Δ resonance which can be estimated by assuming the dominance of S-wave ($L = 0$) or P-wave ($L = 1$) $\Delta \to N\pi$ decay [3,4]:

$$\Gamma_\Delta(W) = \Gamma_\Delta^0 \left(\frac{M_\Delta}{W}\right)^L \left[\frac{p_\pi^\star(W)}{p_\pi^\star(M_\Delta)}\right]^{2L+1},$$

where

$$p_\pi^\star(W) = \sqrt{\left(\frac{W^2 - M^2 + m_\pi^2}{2W}\right)^2 - m_\pi^2},$$

is the pion momentum in the rest frame of the Δ with invariant mass $W = |p'| = |p + q|$; M_Δ and Γ_Δ^0 are, correspondingly, the central mass and decay width of Δ.

The coefficients V_n^{jk}, A_n^{jk} and K_n^{jk} are found to be cubic polynomials in invariant dimensionless variables x, w and parameter $\zeta = M/M_\Delta$. Only 70 among the total 144 coefficients are nonzero (see appendix).

3 Single pion production in Rein–Sehgal model

The Rein-Sehgal (RS) model [5] is one of the most circumstantial and approved phenomenological tools for description of single-pion production through baryon resonances in neutrino and antineutrino interactions with nucleons. It is incorporated into essentially all MC neutrino event generators developed for both accelerator and astroparticle experiments and is in agreement with available experimental data (see, e.g., Ref. [6] for a recent review and further references). However the RS model is not directly applicable to the ν_τ and $\overline{\nu}_\tau$ induced reactions since it neglects the final lepton mass. Due to the same reason, the model is not suited for studying the lepton polarization. In this section, we describe a simple generalization of the RS model based upon the covariant form of the charged leptonic current j_λ with definite lepton helicity discussed in Sect. 2 which allows us to take into account the final lepton mass and spin.

The charged hadronic current in the RS approach has been derived in terms of the Feynman-Kislinger-Ravndal (FKR) relativistic quark model [7] and its explicit form has been written in the resonance rest frame (RRF); below we will mark this frame with asterisk (*). In RRF, the energy of the incoming neutrino, outgoing lepton, target nucleon and the 3-momentum transfer are, respectively,

$$
\begin{aligned}
E_\nu^\star &= \frac{1}{2W}\left(2ME_\nu - Q^2 - m_\ell^2\right) = \frac{E_\nu}{W}\left[M - (E_\ell - P_\ell\cos\theta)\right], \\
E_\ell^\star &= \frac{1}{2W}\left(2ME_\ell + Q^2 - m_\ell^2\right) = \frac{1}{W}\left[ME_\ell + E_\nu(E_\ell - P_\ell\cos\theta) - m_\ell^2\right], \\
E_N^\star &= W - (E_\nu^\star - E_\ell^\star) = \frac{M}{W}(M + E_\nu - E_\ell), \\
|\mathbf{q}^\star| &= \frac{M}{W}|\mathbf{q}| = \frac{M}{W}\sqrt{E_\nu^2 - 2E_\nu P_\ell\cos\theta + P_\ell^2}.
\end{aligned}
\tag{9}
$$

It is convenient to direct the spatial axes of RRF in such a way that $\mathbf{p}^\star = (0, 0, -|\mathbf{q}^\star|)$ and $k_y^\star = k_y'^\star = 0$. These conditions lead to the following system of equations:

$$
\begin{cases}
k_x^\star = k_x'^\star = \sqrt{(E_\nu^\star)^2 - (k_z^\star)^2}, \\
k_z^\star - k_z'^\star = |\mathbf{q}^\star|, \\
k_z^\star + k_z'^\star = \dfrac{1}{|\mathbf{q}^\star|} \left[(E_\nu^\star)^2 - (E_\ell^\star)^2 + m_\ell^2 \right].
\end{cases}
\tag{10}
$$

By using Eqs. (9) and (10) we find the components of the lepton spin 4-vector:

$$
s_0^\star = \frac{1}{m_\ell W} \left[M P_\ell + E_\nu \left(P_\ell - E_\ell \cos\theta \right) \right],
$$

$$
s_x^\star = \frac{E_\nu E_\ell}{m_\ell |\mathbf{q}|} \sin\theta,
$$

$$
s_y^\star = 0,
$$

$$
s_z^\star = \frac{1}{m_\ell |\mathbf{q}| W} \left[\left(E_\nu \cos\theta - P_\ell \right) \left(M E_\ell - m_\ell^2 + E_\nu E_\ell \right) - E_\nu P_\ell \left(E_\nu - P_\ell \cos\theta \right) \right].
$$

Then, by applying the general equations (4), the components of the leptonic current in RRF with the lepton helicity λ measured in lab. frame, are expressed as

$$
j_0^\star = N_\lambda m_\ell \frac{E_\nu}{W} (1 - \lambda \cos\theta) \left(M - E_\ell - \lambda P_\ell \right),
$$

$$
j_x^\star = N_\lambda m_\ell \frac{E_\nu}{|\mathbf{q}|} \sin\theta \left(P_\ell - \lambda E_\nu \right),
$$

$$
j_y^\star = i \lambda N_\lambda m_\ell E_\nu \sin\theta,
$$

$$
j_z^\star = N_\lambda m_\ell \frac{E_\nu}{|\mathbf{q}| W} (1 - \lambda \cos\theta) \left[\left(E_\nu + \lambda P_\ell \right) \left(M - E_\ell \right) + P_\ell \left(\lambda E_\nu + 2 E_\nu \cos\theta - P_\ell \right) \right].
$$

On the other hand, in the spirit of the RS model, the leptonic current still can be treated as the intermediate W boson polarization 4-vector. Therefore it may be decomposed into three polarization 4-vectors corresponding to left-handed, right-handed and scalar polarization:

$$
j_\lambda^\alpha = K^{-1} \left[c_L^\lambda e_L^\alpha + c_R^\lambda e_R^\alpha + c_S^\lambda e_{(\lambda)}^\alpha \right],
$$

$$
e_L^\alpha = \frac{1}{\sqrt{2}} (0, 1, -i, 0),
$$

$$
e_R^\alpha = \frac{1}{\sqrt{2}} (0, -1, -i, 0),
$$

$$
e_{(\lambda)}^\alpha = \frac{1}{\sqrt{Q^2}} \left(\mathcal{Q}_{(\lambda)}^\star, 0, 0, \nu_{(\lambda)}^\star \right).
$$

Here the vectors e_L^α and e_R^α are the same as in Ref. [5] while $e_{(\lambda)}^\alpha$ has been modified to include the lepton mass effect:

$$
\mathcal{Q}_{(\lambda)}^\star = \frac{K \sqrt{Q^2}}{c_S^\lambda} j_0^\star, \quad \nu_{(\lambda)}^\star = \frac{K \sqrt{Q^2}}{c_S^\lambda} j_z^\star, \quad K = \frac{|\mathbf{q}|}{E_\nu \sqrt{2 Q^2}}.
$$

Then the coefficients c_i^λ are explicitly defined through the components j_λ^α in RRF as

$$c_L^\lambda = \frac{K}{\sqrt{2}} \left(j_x^\star + i j_y^\star \right), \quad c_R^\lambda = -\frac{K}{\sqrt{2}} \left(j_x^\star - i j_y^\star \right), \quad c_S^\lambda = K \sqrt{|(j_0^\star)^2 - (j_z^\star)^2|}.$$

For antineutrino induced reactions one have to take into account relation (5). It yields

$$\left[\mathcal{Q}_{(\lambda)}^\star \right]_{\overline{\nu}} = \left[\mathcal{Q}_{(-\lambda)}^\star \right]_\nu, \quad \left[\nu_{(\lambda)}^\star \right]_{\overline{\nu}} = \left[\nu_{(-\lambda)}^\star \right]_\nu,$$

$$\left[c_L^\lambda \right]_{\overline{\nu}} = \lambda \left[c_R^{-\lambda} \right]_\nu, \quad \left[c_R^\lambda \right]_{\overline{\nu}} = \lambda \left[c_L^{-\lambda} \right]_\nu, \quad \left[c_S^\lambda \right]_{\overline{\nu}} = -\lambda \left[c_S^{-\lambda} \right]_\nu.$$

Within the extended RS model, the elements of the polarization density matrix may be written as the superpositions of the partial cross sections[3] $\sigma_L^{\lambda\lambda'}$, $\sigma_R^{\lambda\lambda'}$ and $\sigma_S^{\lambda\lambda'}$:

$$\rho_{\lambda\lambda'} = \frac{\Sigma_{\lambda\lambda'}}{\Sigma_{++} + \Sigma_{--}}, \quad \Sigma_{\lambda\lambda'} = \sum_{i=L,R,S} c_i^\lambda c_i^{\lambda'} \sigma_i^{\lambda\lambda'},$$

and the differential cross section is given by

$$\frac{d^2\sigma}{dQ^2 dW^2} = \frac{G_F^2 \cos^2\theta_C Q^2}{2\pi^2 M |\mathbf{q}|^2} \left(\Sigma_{++} + \Sigma_{--} \right).$$

The partial cross sections are found to be the bilinear superpositions of the reduced amplitudes for producing a $N\pi$ final state with allowed isospin by a charged isovector current:

$$\sigma_{L,R}^{\lambda\lambda'} = \frac{\pi W}{2M} \left(A_{\pm 3}^\lambda A_{\pm 3}^{\lambda'} + A_{\pm 1}^\lambda A_{\pm 1}^{\lambda'} \right),$$

$$\sigma_S^{\lambda\lambda'} = \frac{\pi M |\mathbf{q}|^2}{2WQ^2} \left(A_{0+}^\lambda A_{0+}^{\lambda'} + A_{0-}^\lambda A_{0-}^{\lambda'} \right).$$

The amplitudes for neutrino induced reactions are

$$A_{\varkappa}^\lambda \left(p\pi^+ \right) = \sqrt{3} \sum_{(I=3/2)} a_\varkappa^\lambda (N_3^*),$$

$$A_{\varkappa}^\lambda \left(p\pi^0 \right) = \sqrt{\tfrac{2}{3}} \sum_{(I=3/2)} a_\varkappa^\lambda (N_3^*) - \sqrt{\tfrac{1}{3}} \sum_{(I=1/2)} a_\varkappa^\lambda (N_1^*),$$

$$A_{\varkappa}^\lambda \left(n\pi^+ \right) = \sqrt{\tfrac{1}{3}} \sum_{(I=3/2)} a_\varkappa^\lambda (N_3^*) + \sqrt{\tfrac{2}{3}} \sum_{(I=1/2)} a_\varkappa^\lambda (N_1^*).$$

Here $\varkappa = \pm 3,\ \pm 1,\ 0\pm$ and only those resonances are allowed to interfere which have the same spin and orbital angular momentum as is in the following typical example describing the $n\pi^+$ final state:

$$3 A_\varkappa^\lambda \left(n\pi^+ \right) A_\varkappa^{\lambda'} \left(n\pi^+ \right) =$$

$$\left[\sum a_\varkappa^\lambda \left(S_{31}^+ \right) + \sqrt{2} \sum a_\varkappa^\lambda \left(S_{11}^+ \right) \right] \left[\sum a_\varkappa^{\lambda'} \left(S_{31}^+ \right) + \sqrt{2} \sum a_\varkappa^{\lambda'} \left(S_{11}^+ \right) \right]$$

$$+ \sum_{j=1,3} \left[\sum a_\varkappa^\lambda \left(P_{3j}^+ \right) + \sqrt{2} \sum a_\varkappa^\lambda \left(P_{1j}^+ \right) \right] \left[\sum a_\varkappa^{\lambda'} \left(P_{3j}^+ \right) + \sqrt{2} \sum a_\varkappa^{\lambda'} \left(P_{1j}^+ \right) \right]$$

$$+ \sum_{j=3,5} \left[\sum a_\varkappa^\lambda \left(D_{3j}^+ \right) + \sqrt{2} \sum a_\varkappa^\lambda \left(D_{1j}^+ \right) \right] \left[\sum a_\varkappa^{\lambda'} \left(P_{3j}^+ \right) + \sqrt{2} \sum a_\varkappa^{\lambda'} \left(P_{1j}^+ \right) \right]$$

$$+ \sum_{j=5,7} \left[\sum a_\varkappa^\lambda \left(F_{3j}^+ \right) + \sqrt{2} \sum a_\varkappa^\lambda \left(F_{1j}^+ \right) \right] \left[\sum a_\varkappa^{\lambda'} \left(P_{3j}^+ \right) + \sqrt{2} \sum a_\varkappa^{\lambda'} \left(P_{1j}^+ \right) \right].$$

---

[3]For the reader's convenience we use the same definitions and (almost) similar notation as in Ref. [5].

Any amplitude $a_\varkappa^\lambda(N_i^*)$ referring to one single resonance N_i^* in a definite state of isospin, charge and helicity consists of two factors which describe the production and subsequent decay of the resonance:

$$a_\varkappa^\lambda(N_i^*) = f_\varkappa^\lambda(\nu N \to N_i^*)\, \eta(N_i^* \to N\pi) \equiv f_\varkappa^{\lambda(i)}\, \eta^{(i)}.$$

The decay amplitudes, $\eta^{(i)}$, can be split into three factors,

$$\eta^{(i)} = \mathrm{sign}(\mathrm{N}_i^*)\, \sqrt{\chi_i}\, \eta_{\mathrm{BW}}^{(i)}(\mathrm{W}),$$

irrespective of isospin, charge or helicity of the resonance. Here, the first factor is the decay sign for resonance N_i^* (see Table III of Ref. [5]), χ_i is the elasticity of the resonance taking care of the branching ratio into the πN final state and $\eta_{\mathrm{BW}}^{(i)}(W)$ is the properly normalized Breit-Wigner term with the running width specified by the πN partial wave from which the resonance arises (Eq. (2.31) in Ref. [5]).

The resonance production amplitudes, $f_\varkappa^{\lambda(i)}$, can be calculated within the FKR quark model in exactly the same way as in Ref. [5]. It can be shown that they have the same structure as that given in Table II of Ref. [5] with the only important difference: the three coefficient functions S, B and C involved into the definitions of the amplitudes have to be modified. Indeed, since in our approach the structure of the polarization 4-vector $e_{(\lambda)}^\alpha$ has been changed with respect to that of the original RS model (by including the lepton mass and spin), we have to recalculate its inner products with the vector and axial hadronic currents. To do this, we used the explicit form for the FKR currents given by Ravndal [8]. As a result, the coefficients S, B and C (and thus the resonance production amplitudes) become parametrically dependent of the lepton mass and helicity:

$$S = S^V = \left(\nu_{(\lambda)}^\star \nu^\star - \mathcal{Q}_{(\lambda)}^\star |\mathbf{q}^\star|\right)\left(1 + \frac{Q^2}{M^2} - \frac{3W}{M}\right)\frac{G^V(Q^2)}{6\,|\mathbf{q}|^2},$$

$$B = B^A = \sqrt{\frac{\Omega}{2}}\left(\mathcal{Q}_{(\lambda)}^\star + \nu_{(\lambda)}^\star \frac{|\mathbf{q}^\star|}{aM}\right)\frac{ZG^A(Q^2)}{3W\,|\mathbf{q}^\star|},$$

$$C = C^A = \left[\left(\mathcal{Q}_{(\lambda)}^\star |\mathbf{q}^\star| - \nu_{(\lambda)}^\star \nu^\star\right)\left(\frac{1}{3} + \frac{\nu^\star}{aM}\right)\right.$$
$$\left. + \nu_{(\lambda)}^\star\left(\frac{2}{3}W - \frac{Q^2}{aM} + \frac{n\Omega}{3aM}\right)\right]\frac{ZG^A(Q^2)}{2W\,|\mathbf{q}^\star|}.$$

Here

$$\nu^\star = E_\nu^\star - E_\ell^\star = \frac{M\nu - Q^2}{W}, \quad a = 1 + \frac{W^2 + Q^2 + M^2}{2MW},$$

$G^{V,A}(Q^2)$ are the vector and axial transition form factors and the remaining notation is explained in Ref. [5]. Other 5 coefficients listed in Eq. (3.11) of Ref. [5] are left unchanged.

Acknowledgments

Appendix: Coefficients V_i^{jk}, A_i^{jk}, K_i^{jk}.

Here we list the seventy *nonzero* coefficients V_i^{jk}, A_i^{jk} and K_i^{jk} appearing in Eqs. (8).

$$V_1^{33} = \zeta^3(1-2x)^2w^3 + \zeta[1 + \zeta^2(1-2x)^2]w^2 + 2(1+\zeta)xw,$$
$$V_1^{34} = \zeta^2(1-2x)^2w^3 + [1 - \zeta(2-\zeta)(1-2x)](1-2x)w^2,$$
$$V_1^{35} = \zeta^2(1-2x)w^3 + [1 - \zeta(2-\zeta)(1-2x)]w^2,$$
$$V_1^{44} = \zeta(1-2x)^2w^3 - (1-\zeta)(1-2x)^2w^2,$$
$$V_1^{45} = 2\zeta(1-2x)w^3 - (1-\zeta)(1-4x)w^2,$$
$$V_1^{55} = \zeta w^3 - (1-\zeta)w^2;$$

$$V_2^{33} = 2\zeta^3 xw^2 + 2\zeta(1+\zeta^2)xw,$$
$$V_2^{34} = 2\zeta^2 xw^2 + 2(1-\zeta)^2 xw,$$
$$V_2^{35} = 2\zeta^2(1+2x)xw^2 + 2(1-\zeta)^2 xw,$$
$$V_2^{44} = 2\zeta xw^2 - 2(1-\zeta)xw,$$
$$V_2^{45} = 4\zeta xw^2 - 4(1-\zeta)xw,$$
$$V_2^{55} = 4\zeta^3 x^2 w^3 + 2\zeta[1 - 2\zeta(1-\zeta)x]xw^2 - 2(1-\zeta)xw;$$

$$V_4^{33} = 2\zeta^3(1-x)w^2 + 2\zeta^3(1-x)w - 1 - \zeta,$$
$$V_4^{34} = 2\zeta^2(1-x)w^2 + [1 - 2\zeta(2-\zeta)(1-x)]w,$$
$$V_4^{35} = 2\zeta^2 w^2 - \zeta(2-\zeta)w,$$
$$V_4^{36} = -2\zeta^2 xw^2 - (1-\zeta)^2 w,$$
$$V_4^{44} = 2\zeta(1-x)w^2 - 2(1-\zeta)(1-x)w,$$
$$V_4^{45} = 2\zeta w^2 - 2(1-\zeta)w,$$
$$V_4^{46} = -2\zeta w^2 + 2(1-\zeta)w,$$
$$V_4^{55} = \zeta^3 w^3 - \zeta^2(1-\zeta)w^2,$$
$$V_4^{56} = 2\zeta^3(1-2x)w^3 - 2\zeta\left[1 + \zeta(1-\zeta)(1-2x)\right]w^2 + 2(1-\zeta)w,$$
$$V_4^{66} = \zeta^3(1-2x)^2 w^3 - \zeta\left[\zeta(1-\zeta)(1-2x)^2 - 2x\right]w^2 - 2(1-\zeta)xw;$$

$$V_5^{33} = 2\zeta^3 w^2 + 2\zeta(1+\zeta^2)w,$$
$$V_5^{34} = 2\zeta^2 w^2 + 2(1-\zeta)^2 w,$$
$$V_5^{35} = 2\zeta^2(1+2x)w^2 + 2(1-\zeta)^2 w,$$
$$V_5^{36} = -4\zeta^2 x^2 w^2 - 2(1-\zeta)^2 xw,$$
$$V_5^{44} = 2\zeta w^2 - 2(1-\zeta)w,$$
$$V_5^{45} = 4\zeta w^2 - 4(1-\zeta)w,$$
$$V_5^{46} = -4\zeta xw^2 + 4(1-\zeta)xw,$$
$$V_5^{55} = 4\zeta^3 xw^3 + 2\zeta[1 - 2\zeta(1-\zeta)x]w^2 - 2(1-\zeta)w,$$
$$V_5^{56} = 4\zeta^3(1-2x)xw^3 - 4\zeta[\zeta(1-\zeta)(1-2x) + 1]xw^2 + 4(1-\zeta)xw;$$

$$A_1^{33} = \zeta^3(1-2x)^2w^3 + \zeta[1+\zeta^2(1-2x)^2]w^2 - 2(1-\zeta)xw,$$
$$A_1^{34} = \zeta^2(1-2x)^2w^3 + [1+\zeta(2+\zeta)(1-2x)](1-2x)w^2,$$
$$A_1^{35} = \zeta^2(1-2x)w^2 + [1+\zeta(2+\zeta)(1-2x)]w,$$
$$A_1^{44} = \zeta(1-2x)^2w^3 + (1+\zeta)(1-2x)^2w^2,$$
$$A_1^{45} = 2\zeta(1-2x)w^2 + 2(1+\zeta)(1-2x)w,$$
$$A_1^{55} = \zeta w + 1 + \zeta;$$

$$A_2^{33} = 2\zeta^3xw^2 + 2\zeta(1+\zeta^2)xw,$$
$$A_2^{34} = 2\zeta^2xw^2 + 2(1+\zeta)^2xw,$$
$$A_2^{35} = 2\zeta^2xw,$$
$$A_2^{44} = 2\zeta xw^2 + 2(1+\zeta)xw,$$
$$A_2^{55} = \zeta^3w + \zeta^2(1+\zeta);$$

$$A_4^{33} = 2\zeta^3(1-x)w^2 + 2\zeta^3(1-x)w + 1 - \zeta,$$
$$A_4^{34} = 2\zeta^2(1-x)w^2 + [1+2\zeta(2+\zeta)(1-x)]w,$$
$$A_4^{35} = 2\zeta^2w + \zeta(2+\zeta),$$
$$A_4^{36} = -2\zeta^2xw^2 - (1+\zeta)^2w,$$
$$A_4^{44} = 2\zeta(1-x)w^2 + 2(1+\zeta)(1-x)w,$$
$$A_4^{45} = 2\zeta w + 2(1+\zeta),$$
$$A_4^{46} = -2\zeta w^2 - 2(1+\zeta)w,$$
$$A_4^{55} = \zeta^3w + \zeta^2(1+\zeta),$$
$$A_4^{56} = 2\zeta^3(1-2x)w^2 + 2\zeta[\zeta(1+\zeta)(1-2x) - 1]w - 2(1+\zeta),$$
$$A_4^{66} = \zeta^3(1-2x)^2w^3 + \zeta[\zeta(1+\zeta)(1-2x)^2 + 2x]w^2 + 2(1+\zeta)xw;$$

$$A_5^{33} = 2\zeta^3w^2 + 2\zeta(1+\zeta^2)w,$$
$$A_5^{34} = 2\zeta^2w^2 + 2(1+\zeta)^2w,$$
$$A_5^{35} = 2\zeta^2(1+x)w + (1+\zeta)^2,$$
$$A_5^{36} = -4\zeta^2x^2w^2 - 2(1+\zeta)^2xw,$$
$$A_5^{44} = 2\zeta w^2 + 2(1+\zeta)w,$$
$$A_5^{45} = 2\zeta w + 2(1+\zeta),$$
$$A_5^{46} = -4\zeta xw^2 - 4(1+\zeta)xw,$$
$$A_5^{55} = 2\zeta^3w + 2\zeta^2(1+\zeta),$$
$$A_5^{56} = 2\zeta^3(1-2x)w^2 + 2\zeta^2(1+\zeta)(1-2x)w;$$

$$K_3^{33} = -2\zeta^3(1-2x)^2w^2 + 2\zeta(2-3x)w,$$
$$K_3^{34} = -\zeta^2(1-2x)^2w^2 + 2(1+\zeta)(1-2x)w,$$
$$K_3^{35} = -\zeta^2(1-2x)w + 2(1+\zeta),$$
$$K_3^{43} = -\zeta^2(1-2x)^2w^2 + 2(1-\zeta)(1-2x)w,$$
$$K_3^{44} = \zeta(1-2x)^2w^2,$$
$$K_3^{45} = \zeta(1-2x)w,$$
$$K_3^{53} = -\zeta^2(1-2x)w^2 - 2(1-\zeta)w,$$
$$K_3^{54} = \zeta(1-2x)w^2,$$
$$K_3^{55} = \zeta w.$$

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